Distance Parameters for a Ferrers Graph

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Abstract

A simple graph $G = (V, E)$ is a Ferrers graph if for all distinct $x, y, z, w \in V$ if $xy \in E$ and $zw \in E$ then either $xw \in E$ or$yz \in E$. Since $xy \in E \Leftrightarrow yx \in E$ holds for all simple graphs, the definition of Ferrers graph must be extended to if$xy \in E$ and $zw \in E$ then either $xw \in E$ or $yz \in E$ or $yw \in E$ or $xz \in E$ .It is shown that, for a ferrers graph $d(u,v) \leq 3$ for all vertices $u, v \in V(G)$.

Keywords: Ferrers Graph, Ferrers Tree, Distance, Diameter, Radius.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G v$ or $degv$. A vertex of degree 0 in G is called an *isolated vertex* and a vertex of degree 1 is called a *pendent vertex* or an *end-vertex* of G. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest u –v path in G. If no such path exists (if the vertices lie in different connected components), then the distance is set equal to ∞.The *eccentricity* (v) of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad G* or $r(G)$ and the maximum eccentricity is its *diameter*, *diamG* of G. Two vertices u and v of G are *antipodal* if $d(u, v)$ = *diam G* or$d(G)$.A *bipartite graph G* is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2; (V_1, V_2) is called a *bipartition* of G. If G contains every edge joining V_1 and V_2, then G is called a *complete bipartite graph*. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A *star* is the complete bipartite graph K1,n. A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree*. A non-trivial path is a tree with exactly two end-vertices. A *caterpillar* is a tree of order 3 or more, for which the removal of all

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end-vertices leaves a path. If a relation R over a set A is a Ferrers relation, it holds if aRb and cRd then either aRdor bRc for all distinct elements $a, b, c, d \in A$. Ferrers relation was introduced in [1] for the first time has been utilized for different purposes in extensive and various science fields. The relation was used with concept lattices in formal concept analysis. Some graphs associated by the relation were linked together concept lattices again. A kind of partition presented as Ferrers diagrams. Some preference modeling structures were constructed by using a generalized version of the relation in social choice theory. Throughout the paper we consider G is a connected graph with at least four vertices. Our other notations are standard and taken mainly from [2] and [3].

Observation 1.1. [3]The sum of the degrees of the vertices of a (p, q)-graph G is $2q$.

2. Main Results

In this study, we use a new graph class called Ferrers graph defined by [1] and the author proved, for a Ferrers graph G, $d(u,v) \leq 2$ for all $u,v \in V(G)$. But in our study we proved that'$d(u,v) \leq 3$ for all $u,v \in V(G)$.

Definition 2.1.A simple graph G is a Ferrers graph if for all distinct $x, y, z, w \in V(G)$ if $xy \in E(G)$ and $zw \in E(G)$ then either $xw \in E(G)$or $yz \in E(G)$. since$xy \in E(G) \Leftrightarrow yx \in E(G)$ holds for all simple graphs, the definition of Ferrers graph must be extended to if $xy \in E(G)$and $zw \in E(G)$then either $xw \in E(G)$ or $yz \in E(G)$ or $yw \in E(G)$ or $xz \in E(G)$.

Remark 2.2. Graphs which do not satisfy the above conditions are classified as non-Ferrers graphs. Also, there are graphs which do not have at least four distinct vertices$x, y, z, w \in V(G)$ such that$xy, zw \in E(G)$. That is, the graph does not exist at least two disjoint edges to verify Ferrers condition. This class of graphs is classified as infringe-Ferrers graphs. The obvious examples are C_3 and P_3.

Theorem2.3. If G is a Ferrers graph, then $d(u, v) \leq 3$ for all $u, v \in V(G)$.

Proof. Let G be a Ferrers graph. To prove $d(u, v) \leq 3$. Suppose not, then there exists vertices u, v in G such that $d(u,v) > 3$. Without loss of generality take u,w,x,v be the vertices in G such that $uw, xv \in E(G)$. . Since G is a Ferrers graph, either ux or wx or uv or wv is an edge of G. Since $d(u,v) > 3, uv \notin E(G)$. Suppose$ux \in E(G)$, then $d(u,v) = d(u,x) + d(x,v) = 2$which is impossible. Suppose $wv \in E(G)$, then $d(u,v) = 2$ which is also impossible. Therefore G is not a Ferrers graph. Hence $d(u, v) \leq 3$for all $u,v \in V(G)$.

Corollary2.4. The equality holds in theorem 2.3. For a path $G = P_4$ is Ferrers and $d(u, v) = 3$ where u, v are end vertices of G. Also for a cycle $G = C_4$ is Ferrers, and $d(u, v) = 2$ where u and v are antipodal vertices.

Proof. Consider the graph $G = K_{1,n}$.To prove G is not a Ferrers graph. Suppose not, without loss of generality assume that u, w, x, vbe the distinct vertices in G such that $uw, xv \in E(G)$. Since G is a Ferrers graph, either ux or wxor uvor wv be an edge of G.

Case (i). Suppose$ux \in E(G)$, then we get a path $wuxv$ with two internal vertices, which is a contradiction.

Case (ii). Suppose $wx \in E(G)$, then we get a path $uwxv$ with two internal vertices, which is a contradiction.

Case (iii). Suppose$uv \in E(G)$, then we get a path $wuxv$ with two internal vertices, which is a contradiction.

Case (iv). Suppose$wv \in E(G)$, then we get a path $uwvx$ with two internal vertices, which is a contradiction. Hence in all the cases, G is not a Ferrers graph.

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Remark 2.6. We can easily observe that *G* is an infringe-Ferrers graph as the graph does not contains two disjoint edges**.**

Theorem 2.7. Let *G* is a tree. If *G* is a Ferrers tree if and only if *G* has two internal vertices.

Proof. Let *G* be a tree. Assume that *G* is a Ferrers tree. To prove *G* has two internal vertices.

Case (i). Suppose *G* has only one internal vertex. Since *G* is a tree and has one internal vertex, *G* is either a star or a path containing three vertices. In both the cases *G* is not a Ferrers graph, which is a contradiction to our assumption. Hence *G* has two internal vertices.

Case (ii). Suppose *G* has three or more internal vertices. Since *G* is a tree, and at least three internal vertices, we have *G* is either a tree with diameter greater than three or a path containing five vertices. In both the cases *G* is not a Ferrers graph, which contradicts our assumption. Hence *G* has exactly two internal vertices.

Conversely, Suppose *G* has two internal vertices. To prove that, *G* is a Ferrers tree. Suppose *G* is not a Ferrers tree. Since *G* is a tree, and is not a Ferrers, we have *G* is either a star or *G* is a tree with diameter greater than three or a path containing three vertices. In all the cases *G* does not have two internal vertices. This is not true. Therefore, *G* is a Ferrers tree.

Theorem2.8. If *G* is a Ferrers tree, and then the sum of the degrees of the internal vertices of *G* is equal to the number of vertices of *G*.

Proof. Let G be a Ferrers tree. To prove $\Sigma d(v_i) = |V|$ where v_i's are the internal vertices of G. Since G is a Ferrers tree, by theorem2.7, it has at most two internal vertices. Let $X = \{v_1, v_2\}$ be the set of internal vertices. Let $Y = \{u_1, u_2, \ldots u_n\}$ be the set of end vertices. By Observation 1.1,$\Sigma d(v) = 2q \Rightarrow \Sigma d(v_i) + \Sigma d(u_j) = 2q$ for $i = 1,2$; $j = 1,2,\ldots,n$. Therefore, $d(v_1) + d(v_2) + d(u_1) + d(u_2) + \ldots + d(u_n) = 2q$, where q is the size of G. But for a tree order is p and size is p–1. This implies, $d(v_1) + d(v_2) + d(u_1) + d(u_2) + \ldots + d(u_n) = 2(p-1)$. Therefore, $d(v_1) + d(v_2) = 2(p-1) - n = 2p-(n+2)$ .Hence $\Sigma d(v) = p = |V|$, wherev_i's are the internal vertices of G.

Theorem 2.9. For a Ferrers tree *G*, diam(*G*) = 3 and rad(*G*) = 2.

Proof. Let G be a Ferrers tree. To prove diameter of G is three. Supposediam (G) ≤ 2. Then G is a star or a path containing three vertices or a path containing two vertices. In all the cases G is not a Ferrers tree, which is a contradiction. Hence diam(G) = 3.Now to prove that radius of G is 2. It is enough to prove that $r(G) \neq 3$ and $r(G) \neq 1$.

Case (i). Suppose r(G) = 3, then diam(G) ≥ 3. If diam(G) > 3,then by theorem 2.3, G is not a Ferrers graph, which is a contradiction. Hence r (G) ≠ 3.

Case (ii).Suppose r(G) = 1, then G is a star graph. By theorem 2.5, Star graph is not a Ferrers graph. Hence r (G) ≠ 1.Hence in both the cases r(G) ≠ 3 and r(G) ≠ 1. Thusr(G) = 2.

Theorem 2.10. For a tree G is a Ferrers tree if and only if the internal vertices of G as centres of G. Proof. Suppose G is a Ferrers tree. Then by theorem 2.7, G has two internal vertices say u and v. To prove u and v are centres of G. By theorem 2.9, rad (G) = 2.Therefore, it is enough to prove that e(u) = e(v) =2.If e(u) = e(v) =1,then e is an internal vertex of $k_{1,n}$ which is not Ferrers.If e(u) = e(v) = 3,then u and v are end vertices of G, which is not possible. Hence u and v are the centres of G. Conversely, suppose that the internal vertices of a tree are the centres of G. To prove G is a Ferrers tree. Suppose not, Let u,v,w be the internal vertices of G. Then there exists at least two end vertices say x and y such that ux and wy are edges of G. Without loss of generality take xuvwy is a path containing five vertices. Clearly x and w are the centres of G. Which is a contradiction. Hence G is a Ferrers tree.

Theorem 2.11. Every Ferrers tree G has exactly two centres.

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Proof. Consider a Ferrers tree G. To prove G has exactly two centres. Suppose not. Then G has either one centre or more than two centres.

Case (i). Suppose G has one centre. Then there exists only one vertex $v \in V(G)$such that e (v) =2. This implies v is a pendant vertex of G. But By Theorem 2.10, the internal vertices are its central vertices, which is a contradiction. Therefore G has at least two centres.

Case (ii). Suppose G has more than two centres. Without loss of generality, assume that G has three centres. Let u_1, u_2, u_3 be the centres of G. By theorem 2.9, diam(G) = 3 and rad(G) = 2. Therefore$e(u_1)$ = 2, $e(u_2)$ = 2, and $e(u_3)$ = 2. It is possible only for cycles C_4 and C_5,which contradicts to our assumption that G is a Ferrers tree. Hence G has exactly two centres.

3. Conclusion

In this paper, we found the distance between any two vertices of the Ferrers-graph G such that $d(u,v) \leq 3$ for all $u,v \in V(G)$. We investigated the diameter, radius and centers of a Ferrers tree. Also, we have seen that the centres of a Ferrers tree depend upon the number of vertices especially its internal vertices.

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